Perturbation expansion for particle distributions in hadron storage rings

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Based on the perturbation theory with multiple scales, we have developed a technique to study the evolution of particle distribution as a function of oscillation amplitudes in a hadron storage ring. With a renormalization scheme for the zeroth-order term, a uniformly valid perturbation expansion for the distribution function is obtained. For localized nonlinear perturbations such as the beam-beam interaction at colliding points, this renormalization scheme results in a functional mapping for the particle distribution and the diffusion processes of particles in the beam can be studied numerically without resorting to the tracking of individual particles. A case involving a single nonlinear kick in the ring is presented to illustrate the method in detail.

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I. INTRODUCTION

The understanding of beam dynamics in hadron storage rings has mostly relied on the tracking of a few individual particles. The real beam, however, consists of a large number of particles, typically up to 10^{10} per bunch in large storage rings such as the Superconducting Super Collider (SSC). Consequently it is inconvenient if not impossible to study the real beam behavior by this brute-force tracking which requires many hours of supercomputer CPU time.

Examples of such beam behavior that we are interested in are slow-particle losses and beam-size growth due to either field errors or beam-beam interactions. Observations have shown that the growth of tails of the particle distribution is a serious problem as it enhances the background level in detectors. For large hadron storage rings, the slow growth of the beam emittance is closely related to the long-term behavior of the colliding beam. As beams circulate in the storage ring, particles are gradually lost, and the rate of particle loss and beam-size growth are more important than the stability of individual particles. Therefore a better understanding of these diffusion processes should be based on the study of the multiparticle system.

One way to describe the multiparticle system is through a use of the single-particle distribution in the transverse phase space. In fact, it has already been done for the study of beam dynamics in high-energy electron storage rings but the behavior of the particle distribution in hadron storage rings is not yet understood theoretically. This is due to the different behavior of the particle distribution in the transverse phase space in two cases. For a high-energy electron beam, because of the dominant radiation effect, the time scale for a beam to reach the equilibrium distribution is much less than the storage time. Consequently, the study of the beam dynamics can be focused on the behavior of the distribution near its steady states [1,2]. For a hadron beam, however, the damping time scale is substantially larger than the storage time so that the motion of particles can be described by the Hamiltonian formalism. In the presence of nonlinear perturbations due to either field errors or beam-beam interactions, the particle distribution may not reach any steady state within a fraction of the storage time and this makes it important to study the time evolution of the distribution.

In most cases, the strengths of high-order multipole field errors and beam-beam interactions are small enough to be treated as a perturbation. A straightforward approach is the expansion of the distribution function in powers of the perturbation strength. This method has been used to study the evolution of the distribution with the Fokker-Planck equation [3]. However, it can be shown that this Poincaré-type expansion breaks down in the infinite domain so that the validity of the expansion is limited. In order to obtain a uniformly valid perturbation expansion, we introduce a perturbation expansion with multiple scales [4]. For localized multipole-field errors and beam-beam interactions, this treatment results in a renormalization scheme for the zeroth-order term of the expansion. As a result, the evolution of the distribution on amplitudes can be expressed by a functional mapping.

In this paper we describe the perturbation technique of multiple scales for the study of the evolution of the particle distribution in the transverse phase space in hadron storage rings. This technique can also be applied to the particle distribution in longitudinal phase space. In Sec. II the single-particle distribution function and its Vlasov equation and Fokker-Planck equation are introduced. The failure of the straightforward perturbation expansion is explained in Sec. III. In Sec. IV we establish a multiple-scale expansion for the Vlasov equation and the Fokker-Planck equation. An illustration for this technique is given in Sec. V. In Sec. VI we discuss the Gaussian-distribution approximation which can simplify the numerical computation. Section VII contains summary and final remarks. In what follows, "distribution" refers to the single-particle distribution in the transverse phase space.

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II. EQUATION FOR THE DISTRIBUTION FUNCTION

In terms of action-angle variables $(\vec{I},\vec{\phi})$, the Hamiltonian for betatron motions in a ring of circumference C can be written as

$$H(\vec{I},\vec{\phi},\theta)=\vec{\nu}\cdot\vec{I}+U(\sqrt{2I_1\beta_1}\cos\psi_1,\sqrt{2I_2\beta_2}\cos\psi_2,\theta)\ , \quad (1)$$

where

$$\psi_i=\phi_i-\nu_i\theta+\frac{C}{2\pi}\int_0^\theta\frac{1}{\beta_i}d\theta,\quad i=1,2\ . \quad (2)$$

The independent variable θ is defined as the path length of the reference orbit divided by $C(2/\pi)$. In this paper, vectors are used to denote two-dimensional variables describing the motions in vertical and horizontal planes. Here we assume that there is no linear coupling. $\vec{\beta}(\theta)$ are Courant-Snyder beta functions of the linear lattice. The action-angle variables used here are related to the Cartesian phase-space coordinates (x_i,x_i') by the equations

$$x_i=(2I_i\beta_i)^{1/2}\cos\psi_i\ , \quad (3)$$

$$x_i'=-(2I_i/\beta_i)^{1/2}\left[\sin\psi_i-\frac{\beta_i'}{2}\cos\psi_i\right],\quad i=1,2\ , \quad (4)$$

where the prime denotes derivatives with respect to $(\theta C/2\pi)$. When there is a tune modulation, the transverse tunes $\vec{\nu}$ are a function of θ. The nonlinear perturbation U represents either beam-beam interactions or field errors. We assume that the actions of the nonlinear perturbation can be approximated by kicks in the transverse plane, i.e.,

$$U=\sum_k U^{(k)}(\sqrt{2I_1\beta_1}\cos\psi_1,\sqrt{2I_2\beta_2}\cos\psi_2)\delta_c(\theta-\theta_k)\ , \quad (5)$$

where θ_k are the locations of kicks and

$$\delta_c(\theta-\theta_k)=\sum_n\delta(\theta-\theta_k-2\pi n)\ . \quad (6)$$

Consider a beam consisting of N particles. If we neglect intrabeam collisions, the phase-space distribution of particles can be described by the single-particle distribution $f(\vec{I},\vec{\phi},\theta)$, which satisfies the Vlasov equation

$$\frac{\partial f}{\partial\theta}+\vec{\nu}(\theta)\cdot\frac{\partial f}{\partial\vec{\phi}}=\{U,f\}\ , \quad (7)$$

where $\{\ \}$ is the Poisson bracket.

If the nonlinear perturbation comes from field errors, U is independent of the particle distribution f. In the case of beam-beam interactions with the strong-weak model, the strong beam acts as a nonlinear lens located at collision points. As far as the distribution of weak beam f is concerned, U is also independent of f. In such cases we can rewrite the Hamiltonian Eq. (1) in the form

$$H(\vec{I},\vec{\phi},\theta)=\vec{\nu}\cdot\vec{I}+U_0(\vec{I},\theta)+U_1(\sqrt{2I_1\beta_1}\cos\psi_1,\sqrt{2I_2\beta_{12}}\cos\psi_2,\theta)\ , \quad (8)$$

where U_0 is that part of U which depends on the amplitude only,

$$U_0(\vec{I},\theta)=\langle U(\sqrt{2I_1\beta_1}\cos\psi_1,\sqrt{2I_2\beta_2}\cos\psi_2,\theta)\rangle_{\vec{\phi}}\ , \quad (9)$$

and

$$U_1\equiv U-U_0\ . \quad (10)$$

The Vlasov equation (7) is reduced to

$$\frac{\partial f}{\partial\theta}+\left[\vec{\nu}(\theta)+\frac{\partial U_0}{\partial\vec{I}}\right]\cdot\frac{\partial f}{\partial\vec{\phi}}=\{U_1,f\}=\sum_{i=1}^{2}\left\{-\sqrt{\beta_i}u_i\left[(2I_i)^{-1/2}\cos(\psi_i)\frac{\partial}{\partial\phi_i}+(2I_i)^{1/2}\sin(\psi_i)\frac{\partial}{\partial I_i}\right]\right\}f\ , \quad (11)$$

where $u_i=\partial U_1(x_1,x_2,\theta)/\partial x_i$. Since $H_0=\vec{\nu}\cdot\vec{I}+U_0(\vec{I},\theta)$ is an integrable system, U_1 can be regarded as the perturbation now.

In the case of two strong colliding beams, the distributions $f^{(1)}$ and $f^{(2)}$ of the two beams influence each other according to the Vlasov equation in which the beam-beam perturbation for one beam is a functional of the distribution of the other beam. Two coupled Vlasov equations are given by

$$\frac{\partial f^{(1)}}{\partial\theta}+\vec{\nu}(\theta)\cdot\frac{\partial f^{(1)}}{\partial\vec{\phi}}=\{U[f^{(2)}],f^{(1)}\}=-\sum_{i=1}^{2}\sqrt{\beta_i}\left[(2I_i)^{-1/2}\cos(\psi_i)\frac{\partial f^{(1)}}{\partial\phi_i}+(2I_i)^{1/2}\sin(\psi_i)\frac{\partial f^{(1)}}{\partial I_i}\right]u_i[f^{(2)}] \quad (12)$$

together with another equation with indices 1 and 2 exchanged, and $u_i[f]=\partial U(x_1,x_2,\theta)/\partial x_i$ is a linear functional of f. For a head-on collision,

$$U(\vec{x})=a\int_0^\infty \frac{dq}{q}\int_{-\infty}^{\infty} d\vec{\xi}\,d\vec{\xi}'f(\vec{\xi},\vec{\xi}',\theta)e^{-|\vec{x}-\vec{\xi}|^2q^2}\ , \qquad (13)$$

where a is a constant, $a=4e^2N/(\gamma mc^2)$ for p-p colliders.

In order to express Eqs. (11) and (12) in a single form, we define a functional vector $\mathbf{f}=(f^{(1)},f^{(2)})$ to denote the distribution functions of two colliding beams. Then, the Vlasov equations (11) and (12) can be expressed as

$$\frac{\partial \mathbf{f}}{\partial\theta}+\vec{\omega}\cdot\frac{\partial \mathbf{f}}{\partial\vec{\phi}}=\mathbf{T}\circ\mathbf{f}\ . \qquad (14)$$

For field errors or beam-beam interactions in the strong-weak model, $\mathbf{f}=(f,0)$, $\vec{\omega}(\theta,\vec{I})=\vec{\nu}(\theta)+\partial U_0/\partial\vec{I}$, and $\mathbf{T}\circ\mathbf{f}=\{U_1,f\}$. For beam-beam interactions in the strong-strong model, $\vec{\omega}(\theta)=\vec{\nu}(\theta)$ and $\mathbf{T}f^{(i)}=\{U[f^j],f^{(i)}\}$ with $i,j=1,2,\ j\neq i$.

If we consider noise and damping, the equation of the distribution is described by the Fokker-Planck equation [5]

$$\frac{\partial \mathbf{f}}{\partial\theta}+\mathbf{L}_{\mathrm{FP}}\circ\mathbf{f}=\mathbf{T}\circ\mathbf{f}\ , \qquad (15)$$

where $\mathbf{T}\circ\mathbf{f}=\{U,\mathbf{f}\}$ and $\mathbf{L}_{\mathrm{FP}}$ is the linear Fokker-Planck operator. In normalized variables,

$$\mathbf{L}_{\mathrm{FP}}=\sum_{i=1}^{2}\left\{\frac{\partial}{\partial\xi_i}\nu_i\eta_i+\frac{\partial}{\partial\eta_i}(-2\alpha_i\eta_i-\nu_i\xi_i)-D_i\frac{\partial^2}{\partial\eta_i^2}\right\}\ , \qquad (16)$$

where $\vec{\alpha}$ and $\vec{D}$ are damping and diffusion coefficients, respectively. When $\vec{\alpha}=\vec{0}$ and $\vec{D}=\vec{0}$, Eq. (15) reduces to the Vlasov equation (14) with $\mathbf{L}_{\mathrm{FP}}=\vec{\nu}(\theta)$.

If we know the evolution of the distribution by solving the Vlasov equation (14) or the Fokker-Planck equation (15), the rms beam size can be evaluated from

$$\langle\vec{I}\rangle=\int\vec{I}f\,d\vec{I}\,d\vec{\phi}=\int\vec{I}\langle f\rangle_{\vec{\phi}}d\vec{I}\ , \qquad (17)$$

where $\langle\ \rangle_{\vec{\phi}}$ denotes the integral with respect to $\vec{\phi}$. In general, neither the Vlasov equation nor the Fokker-Planck equation can be solved exactly for the nonlinear system of Eq. (1). By inspecting Eq. (17), however, we see that the particle distribution on amplitudes alone is required for our purpose; the Vlasov equation and the Fokker-Planck equation contain more information than is needed. By removing the unnecessary information, the problem may be simplified and easily handled with perturbation methods.

III. STRAIGHTFORWARD PERTURBATION EXPANSION

In many practical problems, the nonlinear perturbation on the right-hand side of Eqs. (14) and (15) can be treated as small in some sense, i.e., the strength of the perturbation can be used as a small parameter for the perturbation expansion. Thus we assume that the distribution function can be expanded as

$$\mathbf{f}(\vec{I},\vec{\phi},\theta)=\sum_{m=0}^{\infty}\epsilon^m\mathbf{f}_m(\vec{I},\vec{\phi},\theta)\ , \qquad (18)$$

where $\epsilon\sim\|\vec{U}\|$, and, for an initial distribution $\mathbf{f}(\vec{I},\vec{\phi},0)=\mathbf{f}_{\mathrm{in}}(\vec{I},\vec{\phi})$, the initial conditions of $\mathbf{f}_m$ are

$$\mathbf{f}_0(\vec{I},\vec{\phi},0)=\mathbf{f}_{\mathrm{in}}(\vec{\mathbf{I}},\vec{\phi})\ , \qquad (19)$$

$$\mathbf{f}_m(\vec{I},\vec{\phi},0)=\mathbf{0}\quad\text{for } m>0\ . \qquad (20)$$

In the case of beam-beam interactions in the strong-strong model, because of the linear dependence of U on the distribution [see Eq. (13)], $\mathbf{T}$ can also be expanded as

$$\mathbf{T}=\sum_{m=0}^{\infty}\epsilon^m\mathbf{T}_m\ , \qquad (21)$$

where $\mathbf{T}_m=\{U[\mathbf{f}_m],\ \}$. For field errors or beam-beam interactions in the strong-weak model, $\mathbf{T}_0=\{U,\ \}$ ($\mathbf{T}_0=\{U_1,\ \}$ for the Vlasov equation) and $\mathbf{T}_m=0$ for $m>0$.

Substituting Eqs. (18) and (21) into Eq. (15) and equating coefficients of equal powers of ϵ to zero, we obtain

$$\frac{\partial\mathbf{f}_0}{\partial\theta}+\mathbf{L}_{\mathrm{FP}}\circ\mathbf{f}_0=0\ , \qquad (22)$$

$$\frac{\partial\mathbf{f}_1}{\partial\theta}+\mathbf{L}_{\mathrm{FP}}\circ\mathbf{f}_1=\mathbf{T}_0\circ\mathbf{f}_0\ , \qquad (23)$$

$$\frac{\partial\mathbf{f}_2}{\partial\theta}+\mathbf{L}_{\mathrm{FP}}\circ\mathbf{f}_2=\mathbf{T}_0\circ\mathbf{f}_1+\mathbf{T}_1\circ\mathbf{f}_0 \qquad (24)$$

$$\cdots$$

Since for any function f

$$\int(\mathbf{T}_qf)d\vec{\phi}\,d\vec{I}=\mathbf{0},\quad\forall q\ , \qquad (25)$$

the normalization condition of the distribution

$$\int d\vec{I}\,d\vec{\phi}\,f(\vec{I},\vec{\phi},\theta)=\int d\vec{I}\,d\vec{\phi}\,f(\vec{I},\vec{\phi},0)=1 \qquad (26)$$

is guaranteed in this expansion.

After all $\mathbf{f}_i$'s for $i<(q-1)$ are known, the qth-order equation for $\mathbf{f}_q$ takes the form

$$\frac{\partial\mathbf{f}_q}{\partial\theta}+\mathbf{L}_{\mathrm{FP}}\circ\mathbf{f}_q=\mathbf{F}(\vec{I},\vec{\phi},\theta)\ . \qquad (27)$$

For the initial condition $\mathbf{f}_q(\vec{I},\vec{\phi},0)$ in Eqs. (19) and (20), the solution of Eq. (27) is

$$\begin{aligned}f_q(\vec{I},\vec{\phi},\theta)=&\int d\vec{I}'d\vec{\phi}'G_2(\vec{I},\vec{I}',\vec{\phi}-\vec{\phi}',\theta)\mathbf{f}_q(\vec{I}',\vec{\phi}',0)\\&+\int d\theta'\int d\vec{I}'d\vec{\phi}'G_2(\vec{I},\vec{I}',\vec{\phi}-\vec{\phi}',\theta-\theta')\\&\times\mathbf{F}(\vec{I}',\vec{\phi}',\theta')\ ,\end{aligned} \qquad (28)$$

where G_2 is the Green's function of Eq. (27) [5],

$$\begin{aligned}G_2(\vec{I},\vec{I}'\vec{\phi}-\vec{\phi}',\theta-\theta')=&G_1(I_1,I_1',\phi_1-\phi_1',\theta-\theta')\\&\times G_1(I_2,I_2',\phi_2-\phi_2',\theta-\theta')\ ,\end{aligned} \qquad (29)$$

with

$$G_1(I_i,I_i',\phi_i-\phi_i',\theta-\theta')=\frac{A_i^{1/2}}{\pi}\exp\{-2A_i[I_i+I_i'e^{-\alpha_i(\theta-\theta')}-2(I_iI_i'e^{-\alpha_i(\theta-\theta')})^{1/2}\cos\tilde{\phi}_i]\}\ , \quad (30)$$

where

$$A_i=\frac{\alpha_i}{D_i(1-e^{-2\alpha_i(\theta-\theta')})}\ ,$$
$$\tilde{\phi}_i=\phi_i-\int_{\theta'}^{\theta}\nu_i(\tau)d\tau-\phi_i'\ . \quad (31)$$

When $\vec{\alpha}=\vec{0}$ and $\vec{D}=\vec{0}$,

$$G_2(\vec{I},\vec{I}',\vec{\phi}-\vec{\phi}',\theta-\theta')=\delta(\vec{I}-\vec{I}')\delta\left[\vec{\phi}-\int_{\theta'}^{\theta}\vec{\omega}(\tau)d\tau-\vec{\phi}'\right]\ , \quad (32)$$

and Eq. (28) reduces to

$$\mathbf{f}_q(\vec{\mathbf{I}},\vec{\phi},\theta)=\mathbf{f}_q\left[\vec{I},\vec{\phi}-\int_0^{\theta}\vec{\omega}(\tau)d\tau,0\right]+\int_0^{\theta}\mathbf{F}\left[\vec{I},\vec{\phi}-\int_{\tau_1}^{\theta}\vec{\omega}(\tau_2)d\tau_2,\tau_1\right]d\tau_1\ . \quad (33)$$

By solving these expansion equations order by order, we obtain a truncated sequence of $\mathbf{f}$. Since

$$\langle \mathbf{f}_q(\vec{I},\vec{\phi},\theta)\rangle_{\vec{\phi}}=\int d\vec{I}'\langle G_2(\vec{I},\vec{I}',\vec{\phi},\theta)\rangle_{\vec{\phi}}\langle \mathbf{f}_q(\vec{I}',\vec{\phi},0)\rangle_{\vec{\phi}}+\int d\theta'\int dI'\langle G_2(\vec{I},\vec{I}',\theta-\theta')\rangle_{\vec{\phi}}\times\langle \mathbf{F}(\vec{I}',\vec{\phi},\theta')\rangle_{\vec{\phi}}\ , \quad (34)$$

if there is a secular term $\langle F\rangle_{\vec{\phi}}\neq\mathbf{0}$, $\mathbf{f}_q$ is proportional to θ for the perturbation in Eq. (5). Consequently, the straightforward expansion of Eq. (18) breaks down when $\theta\sim O(\epsilon^{-1})$ since, beyond that, the condition for a uniform asymptotic sequence $|f_q^{(i)}/f_{q-1}^{(i)}|<\infty$ for $i=1,2$ cannot be satisfied. With Eqs. (22)–(24) it can be easily shown that the secular terms are generic in these expansion equations. Thus the straightforward expansion is not valid for our problem. As a matter of fact the appearance of the secular terms is a characteristic of nonlinear problems. In order to obtain a proper perturbation expansion, these secular terms must be eliminated systematically.

IV. MULTIPLE-SCALE EXPANSION

Equation (17) shows that for the study of particle loss and beam-size growth, only the particle distribution on amplitudes is required. Thus the average with respect to phase $\vec{\phi}$ can be utilized to cure the nonuniform problem due to the secular terms. In order to obtain a truncated expansion valid for all times up to $O(\epsilon^{-M})$, we introduce a set of multiple time scales $t_0,t_1,\ldots,t_M$, where

$$t_m=\epsilon^m\theta,\quad m=0,1,\ldots,M\ . \quad (35)$$

In general, the time scale t_m is slower than t_{m-1}. Instead of using the expansion (18), we assume that

$$\mathbf{f}(\vec{I},\vec{\phi},\theta)=\mathbf{f}(\vec{I},\vec{\phi},t_0,t_1,\ldots,t_M)=\sum_{m=0}^{M-1}\epsilon^m\mathbf{f}_m(\vec{I},\vec{\phi},t_0,t_1,\ldots,t_M)+O(\epsilon^M)\ . \quad (36)$$

Here the truncated expansion is assumed to be valid only for times up to $O(\epsilon^{-M})$. Beyond that, other time scales must be included to keep the expansion uniformly valid. By using the chain rule, the derivative with respect to θ can be transformed into the derivatives with respect to $\{t_m\}$ according to

$$\frac{\partial}{\partial\theta}=\sum_{m=0}^{M}\epsilon^m\frac{\partial}{\partial t_m}\ . \quad (37)$$

Substituting Eqs. (36), (37), and (21) into Eq. (15) and equating coefficients of equal powers of ϵ on both sides, we obtain

$$\frac{\partial f_0}{\partial t_0}+\mathbf{L}_{\rm FP}\circ\mathbf{f}_0=0\ , \quad (38)$$

$$\frac{\partial \mathbf{f}_1}{\partial t_0}+\frac{\partial \mathbf{f}_0}{\partial t_1}+\mathbf{L}_{\rm FP}\circ\mathbf{f}_1=\mathbf{T}_0\circ\mathbf{f}_0\ , \quad (39)$$

$$\frac{\partial \mathbf{f}_2}{\partial t_0}+\frac{\partial \mathbf{f}_1}{\partial t_1}+\frac{\partial \mathbf{f}_0}{\partial t_2}+\mathbf{L}_{\rm FP}\circ\mathbf{f}_2=\mathbf{T}_0\circ\mathbf{f}_1+\mathbf{T}_1\circ\mathbf{f}_0\ . \quad (40)$$

Since Eq. (25) is still valid here, the normalization condition Eq. (26) is also guaranteed in the expansion (36). Now, in the qth-order equation, we can choose

$$\sum_{m=1}^{q}\frac{\partial \mathbf{f}_{q-m}}{\partial t_m}=\left\langle\sum_{m=0}^{q-1}\mathbf{T}_m\mathbf{f}_{q-m-1}\right\rangle_{\vec{\phi}} \quad (41)$$

to eliminate the secular terms. As q is greater than one, the dependence of $\mathbf{f}_m$ on $\{t_m|m=1,\ldots,q\}$ is underdetermined so that we can select a particular set of equations

$$\frac{\partial \mathbf{f}_0}{\partial t_q}=\left\langle\sum_{m=0}^{q-1}\mathbf{T}_m\mathbf{f}_{q-m-1}\right\rangle_{\vec{\phi}}\ , \quad (42)$$

$$\frac{\partial \mathbf{f}_i}{\partial t_j}=0\quad\text{for } i,j>0\ . \quad (43)$$

Consequently, $\langle\mathbf{f}_m\rangle_{\vec{\phi}}=\mathbf{0}$ for $m>0$, and the zeroth-order term $\mathbf{f}_0$ is renormalized by including in it all phase-independent parts of the distribution function. If the initial distribution depends on amplitudes only,

$$\langle\mathbf{f}\rangle_{\vec{\phi}}=\sum_{m=0}^{\infty}\epsilon^m\langle\mathbf{f}_m\rangle_{\vec{\phi}}=\mathbf{f}_0 \quad (44)$$

so that $\mathbf{f}_0$ contains all the information needed for the study of particle losses and beam-size growth.

V. AN ILLUSTRATION

To illustrate this theory, we consider a simple case in which $\vec{\alpha}=\vec{0}$, $\vec{D}=\vec{0}$, and there is only one nonlinear kick located at $\theta=0$, either due to multipole-field errors or beam-beam interaction in the strong-weak model,

$$U=U^{(0)}(\sqrt{2I_1\beta_1}\cos\phi_1,\sqrt{2I_2\beta_2}\cos\phi_2)\delta_c(\theta) . \quad (45)$$

The initial distribution is assumed to be phase independent,

$$f(\vec{I},\vec{\phi},0)=f_{\rm in}(\vec{I}) . \quad (46)$$

From Eq. (38) we have

$$f_0=f_0(\vec{I},t_1,t_2,\ldots) . \quad (47)$$

Inserting this result and Eq. (45) into Eq. (39), we get

$$\frac{\partial f_1}{\partial t_0}+\frac{\partial f_0}{\partial t_1}+\vec{\omega}(\theta)\cdot\frac{\partial f_1}{\partial\vec{\phi}} =-\sum_{i=1}^{2}(2\beta_iI_i)^{1/2}\sin\phi_i\frac{\partial f_0}{\partial I_i}u_i\delta_c(\theta) , \quad (48)$$

where

$$u_i(\sqrt{2I_1\beta_1}\cos\phi_1,\sqrt{2I_2\beta_2}\cos\phi_2)=\partial U^0(x_1,x_2)/\partial x_i ,$$

for $i=1,2$. Since

$$\langle u_i\sin\phi_i\rangle_{\vec{\phi}}=0 , \quad (49)$$

there is no secular term in Eq. (48), $\partial f_0/\partial t_1=0$, and $f_0=f_0(\vec{I},t_2,\ldots)$. Thus, in the first-order perturbation, $f_0(\vec{I})=f_{\rm in}(\vec{I})$, i.e., there is no beam-size growth in the first-order perturbation and we must consider the second-order term. From Eq. (33), f_1 is obtained as

$$f_1=-\sum_{n=0}^{[\theta/2\pi]}\sum_{i=1}^{2}(2\beta_iI_i)^{1/2}\sin(\phi_{ni})\frac{\partial f_0}{\partial I_i}u_i(n) , \quad (50)$$

where $[\theta/2\pi]$ is the integer part of $\theta/2\pi$,

$$\vec{\phi}_n=(\phi_{n1},\phi_{n2})=\vec{\phi}-\int_{2\pi n}^{\theta}\vec{\omega}(\tau)d\tau , \quad (51)$$

and

$$u_i(n)=u_i(\sqrt{2I_1\beta_1}\cos\phi_{n1},\sqrt{2I_2\beta_2}\cos\phi_{n2}) .$$

Substituting f_0 and f_1 into Eq. (40) and using Eq. (42) to eliminate the secular terms, we have

$$\frac{\partial f_0}{\partial t_2}=\langle\{U^{(0)},f_1\}\rangle_{\vec{\phi}}\delta_c(\theta) . \quad (52)$$

Since we did not specify the strength of multipole-field errors or the parameter of beam-beam interaction separately from the perturbation $U^{(0)}$, ϵ is used here simply to track the order of the perturbation expansion. In the final expansion, ϵ should be taken as unity. To the second order in the strength of U, we have

$$f_0^{M+1}=f_0^M+\langle\{U^{(0)},f_1^M\}\rangle_{\vec{\phi}} , \quad (53)$$

with

$$\begin{aligned}\langle\{U^{(0)},f_1^M\}\rangle_{\vec{\phi}}=\sum_{n=0}^{M}\sum_{j=1}^{2}\Bigg\{&\beta_j\frac{\partial f_0^M}{\partial I_j}\cos(\mu_{nj})\langle u_j(n)u_j\rangle_{\vec{\phi}}\\ &+\sum_{i=1}^{2}\frac{\partial f_0^M}{\partial I_i}\Bigg[\beta_j\sqrt{2\beta_iI_i}\sin(\mu_{nj})\langle\sin(\phi_{ni})u_{ij}(n)u_j\rangle_{\vec{\phi}}\\ &\qquad+\sqrt{2\beta_iI_i}\sqrt{2\beta_jI_j}D_{ij}\langle\sin(\phi_j)\cos(\phi_{nj})u_i(n)u_j\rangle_{\vec{\phi}}\\ &\qquad-\sum_{k=1}^{2}\sqrt{2\beta_iI_i}\sqrt{2\beta_jI_j}\sqrt{2\beta_kI_k}D_{jk}\langle\sin(\phi_j)\sin(\phi_{ni})\sin(\phi_{nk})u_{ik}(n)u_j\rangle_{\vec{\phi}}\Bigg]\\ &+\sum_{i=1}^{2}\sqrt{2\beta_iI_i}\sqrt{2\beta_jI_j}\frac{\partial^2f_0^M}{\partial I_i\partial I_j}\langle\sin(\phi_j)\sin(\phi_{ni})u_i(n)u_j\rangle_{\vec{\phi}}\Bigg\} ,\end{aligned} \quad (54)$$

where $\vec{\mu}_n=\int_{2\pi n}^{\theta}\vec{\omega}(\tau)d\tau$, $D_{ij}=-(\partial U_0/\partial I_i\partial I_j)(\theta-2\pi n)$, and $u_{ij}=\partial u_i/\partial x_j$. $M=[\theta/2\pi]$ is the number of turns and the superscript M denotes the distribution function after M turns. Equation (53) is a functional mapping for the evolution of the particle distribution on amplitudes. For a given initial distribution, the evolution of the distribution on amplitudes as well as the evolution of beam size can be studied by iterating this map with computer.

From Eq. (53), the physical meaning of the technique of multiple scales is quite clear. Each time the beam passes through a weak nonlinear kick, its distribution is perturbed slightly. However, only the phase-independent part of this change in the distribution is relevant to the problem of particle loss and beam-size growth. Therefore after each kick we use this phase-independent part of the change in the distribution to renormalize the zeroth-order term of its expansion, f_0. The phase-dependent part is eliminated by taking average over $\vec{\phi}$. A uniformly valid perturbation expansion is therefore obtained by reexpanding the distribution with this update zeroth-order term after each kick. This renormalization scheme results in the functional mapping (53) in which only action variables $\vec{I}$ are involved. In comparison with the direct simulation of particle distribution in phase space [6], this scheme greatly simplifies the numerical computation.

It should be pointed out that the result to the second order in the strength of U is only valid for the "time" up to $O(\|U\|^{-2})$. For the beam-beam interaction, the beam-beam parameter is typically of the order of 10^{-3} in larger colliders such as SSC. This corresponds to 10^5-10^6 turns and, beyond that, the next order terms must be taken into account.

In order to check the formalism, we consider a beam near the difference resonance of a single sextupole kick, i.e.,

$$\nu_1-2\nu_2=l \ , \tag{55}$$

where $l\geq 0$ is an integer. The nonlinear perturbation of the Hamiltonian can be approximated by the resonance Hamiltonian

$$U^{(0)}=-\epsilon I_1^{1/2}I_2\cos(\phi_1-2\phi_2)\delta_c(\theta) \ . \tag{56}$$

Since $2I_1+I_2$ is a constant of the motion for $U^{(0)}$ of Eq. (56), the rms beam sizes in two directions also satisfy

$$2\langle I_1\rangle+\langle I_2\rangle=\text{const} \ . \tag{57}$$

Assume that the beam is a round Gaussian beam initially. After substituting Eq. (56) into Eq. (53), the evolution of rms beam sizes, as shown in Fig. 1, and the change of the particle distribution on amplitudes, as shown in Fig. 2, were easily obtained. It can be seen that Eq. (57) is indeed held here. Figure 1 also shows a good agreement between the result of the mapping (53) and the result of multiparticle tracking [7].

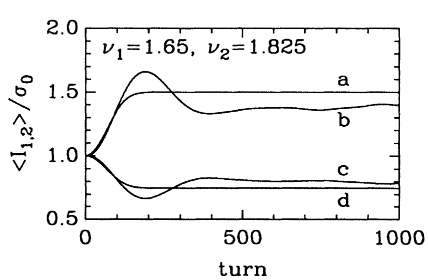

FIG. 1. The evolution of rms beam sizes when the beam is on a difference resonance. The beam is a round Gaussian beam with a rms beam size σ_0 initially. The upper two curves are $\langle I_2\rangle$ and the lower two $\langle I_1\rangle$. Curves a and d are the result of the map (53) with the nonlinear perturbation of Eq. (56). Curves b and c are from the tracking of 5000 particles with a single sextupole kick, $\epsilon\sigma_0^{1/2}=0.001$.

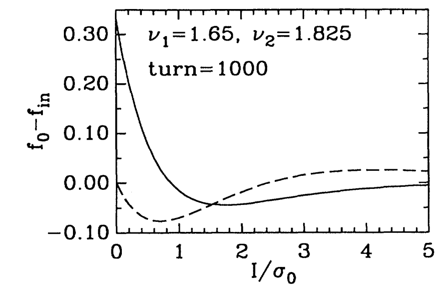

FIG. 2. The change in the particle distribution as a function of the amplitudes for the same case as in Fig. 1. The solid (dashed) curve is the difference between the distribution on I_1 (I_2) at 1000 turns and its initial distribution.

VI. GAUSSIAN-DISTRIBUTION APPROXIMATION

Experimental observations show that the particle distribution in hadron colliders remains approximately Gaussian if it is initially Gaussian [8]. As the beam circulates in the ring, this distribution is gradually distorted with a growth of the distribution tail. In this case we can further simplify the functional mapping (53) for numerical computation by approximating f_0^M in Eq. (54) as a Gaussian distribution

$$f_0^M\simeq\frac{1}{(2\pi)^2\sigma_1^M\sigma_2^M}\exp\left[-\frac{I_1}{\sigma_1^M}-\frac{I_2}{\sigma_2^M}\right] , \tag{58}$$

where

$$\vec{\sigma}^M=\int\vec{I}f_0^M d\vec{I} \ . \tag{59}$$

The mapping for the evolution of the distribution on amplitudes can be greatly simplified for the numerical computation, since we now have

$$\frac{\partial f_0^M}{\partial I_i}=-\frac{1}{\sigma_i^M}f_0^M \ , \tag{60}$$

$$\frac{\partial^2 f_0^M}{\partial I_i\partial I_j}=\frac{1}{\sigma_i^M\sigma_j^M}f_0^M \ . \tag{61}$$

VII. SUMMARY

Using a perturbation expansion with multiple scales, we have solved the Vlasov equation and the Fokker-Planck equation in the time domain for the particle distribution in hadron storage rings. In order to eliminate secular terms in the expansion, we renormalize the zeroth-order term of the expansion by including in it the phase-independent part of the perturbed distribution. The phase-dependent part of the distribution function is eliminated by taking average with respect to angle variables. As a result, the zeroth-order term represents the particle distribution of amplitudes. For localized nonlinear perturbations, this renormalization scheme is reduced to a functional mapping for the evolution of the distribution of amplitudes. In this mapping, only action variables are involved so that the evolution of the particle distribution and the beam size can be easily studied by numerically iterating the mapping. One advantage of this method is that we can treat beam-beam interactions in a self-consistent manner.

When the system is close to major resonances, the perturbation expansion of the distribution function may not converge. A typical example is the resonant extraction. In this case, the beam is strongly perturbed and the particle distribution in phase space is determined predominantly by a single, low-order resonance. The perturbed distribution may deviate too far from its unperturbed form to be considered within the framework of perturbation theory. Furthermore, in the resonant extraction situation, the distribution in phase is as important as the distribution in amplitude. Therefore the perturbation expansion involving the averaging over phase will emasculate the physics in such a situation. Large storage rings, however, are generally operated far from all major resonances. The important problem of our concern here is the slow-particle loss and the beam-size growth due to weak nonlinear perturbation, which comes primarily from high-order resonances, and the description based on the particle distribution is more meaningful.

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